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An Extension of Enestrom-Kakeya Theorem

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Abstract

In this paper we give an extension of the famous Enestrom-Kakeya Theorem, which generalizes many generalizations of the said theorem as well.

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Introduction

A famous result giving a bound for all the zeros of a polynomial with real positive monotonically decreasing coefficients is the following result known as Enestrom-Kakeya theorem [4]:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then all the zeros of P(z) lie in the closed disk $|z| \le 1$.

If the coefficients are monotonic but not positive, Joyal, Labelle and Rahman [3] gave the following generalization of Theorem A:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0.$$

Then all the zeros of P(z) lie in the closed disk $|z| \le \frac{a_n - a_0 + |a_0|}{|a_n|}$.

Aziz and Zargar [1] generalized Theorem B by proving the following result:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0.$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k-1| \le \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

Gulzar [2] generalized Theorem C to polynomials with complex coefficients and proved the following results:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$,

$\mathrm{Im}(a_j)=\beta_j, j=0,1,......,n$ such that for some $k \ge 1, 0<\tau \le$,

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$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$\left| z + (k-1)\frac{\alpha_n}{a_n} \right| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|} .$$

Theorem E: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j, j = 0,1,......,n$ such that for some $k \geq 1, 0 < \tau \leq$,

$$k\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \tau\beta_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$\left| z + (k-1)\frac{\beta_n}{a_n} \right| \leq \frac{k\beta_n + 2|\beta_0| - \tau(\beta_0 + |\beta_0|) + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|} .$$

Recently, Liman and Shah [5] proved the following generalization of Theorem C for polynomials having real coefficients:

Theorem F: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients such that for some t>0 and $1 \leq \lambda \leq n$,

$$a_0 \leq a_1 \leq ...... \leq a_{\lambda-1} \leq ta_\lambda \leq t^2 a_{\lambda+1} \leq ...... \leq t^{n-\lambda} a_{n-1} \leq t^{n-k+1} a_n .$$

Then all the zeros of P(z) lie in

$$|z + t - 1| \leq \frac{a_n - a_0 + |a_0| + (t-1)\{\sum_{j=\lambda}^{n}(a_j + |a_j|) - |a_n|\}}{|a_n|} .$$

The aim of this paper is to apply Theorem F to polynomials with complex coefficients and prove

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j, j = 0,1,......,n$ such that for some t>0 , $k \geq 1, 0 < \tau \leq 1$ and $1 \leq \lambda \leq n$,

$$\tau\alpha_0 \leq \alpha_1 \leq ...... \leq \alpha_{\lambda-1} \leq t\alpha_\lambda \leq t^2\alpha_{\lambda+1} \leq ...... \leq t^{n-\lambda}\alpha_{n-1} \leq kt^{n-k+1}\alpha_n .$$

Then all the zeros of P(z) lie in

$$\left| z + (kt-1)\frac{\alpha_n}{a_n} \right| \leq \frac{kt\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + (t-1)\sum_{j=\lambda}^{n-1}(\alpha_j + |\alpha_j|) + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|} .$$

If a_j is real i.e. $\beta_j = 0, \forall j = 0,1,......,n$, we immediately get the following result:

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some t>0 , $k \geq 1, 0 < \tau \leq 1$ and $1 \leq \lambda \leq n$,

$$\tau a_0 \leq a_1 \leq ...... \leq a_{\lambda-1} \leq ta_\lambda \leq t^2 a_{\lambda+1} \leq ...... \leq t^{n-\lambda} a_{n-1} \leq kt^{n-k+1} a_n .$$

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Then all the zeros of P(z) lie in

$$|z+kt-1| \le \frac{kta_n + 2|a_0| - \tau(a_0+|a_0|) + (t-1)\sum_{j=\lambda}^{n-1}(a_j+|a_j|)}{|a_n|}.$$

Remark 1: For k=1, $\tau=1$, Cor. 1 reduces to Theorem F. For t=1, Theorem 1 reduces to Theorem D.

Applying Theorem 1 to the polynomial –iP(z), we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j, j=0,1,......,n$ such that for some t>0 , $k\ge 1, 0<\tau\le 1$ and $1\le\lambda\le n$,

$$\tau\beta_0 \le \beta_1 \le ...... \le \beta_{\lambda-1} \le t\beta_\lambda \le t^2\beta_{\lambda+1} \le ...... \le t^{n-\lambda}\beta_{n-1} \le kt^{n-k+1}\beta_n.$$

Then all the zeros of P(z) lie in

$$|z+kt-1| \le \frac{kt\beta_n + 2|\beta_0| - \tau(\beta_0+|\beta_0|) + (t-1)\sum_{j=\lambda}^{n-1}(\beta_j+|\beta_j|) + 2\sum_{j=0}^{n}|\alpha_j|}{|a_n|}.$$

Remark 2: For t=1, Theorem 1 reduces to Theorem E.

Taking k=1, we get the following result from Theorem 1:

Corollary 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j, j=0,1,......,n$ such that for some t>0 , $0<\tau\le 1$ and $1\le\lambda\le n$,

$$\tau\alpha_0 \le \alpha_1 \le ...... \le \alpha_{\lambda-1} \le t\alpha_\lambda \le t^2\alpha_{\lambda+1} \le ...... \le t^{n-\lambda}\alpha_{n-1} \le t^{n-k+1}\alpha_n.$$

Then all the zeros of P(z) lie in

$$\left|z+(t-1)\frac{\alpha_n}{a_n}\right| \le \frac{t\alpha_n + 2|\alpha_0| - \tau(\alpha_0+|\alpha_0|) + (t-1)\sum_{j=\lambda}^{n-1}(\alpha_j+|\alpha_j|) + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

For other different values of the parameters in the above results, we get many other interesting results.

Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

F(z)=(1-z)P(z)

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_{\lambda+1}z^{\lambda+1} + a_\lambda z^\lambda + a_{\lambda-1}z^{\lambda-1} + ... + a_{n-1}z^{n-1} + a_n z^n$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1}-a_{n-2})z^{n-1} + ... + (a_{\lambda+1}-a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda$$
$$+ (a_{\lambda-1}-a_{\lambda-2})z^{\lambda-1} + ... + (a_1-a_0)z + a_0$$

$$= -a_n z^{n+1} + \{(kt\alpha_n - \alpha_{n-1}) - (kt\alpha_n - \alpha_n)\}z^n + \{(t\alpha_{n-1}-\alpha_{n-2}) - (t\alpha_{n-1}-\alpha_{n-1})\}z^{n-1}$$
$$+ ...... + \{(t\alpha_{\lambda+1}-\alpha_\lambda) - (t\alpha_{\lambda+1}-\alpha_{\lambda+1})\}z^{\lambda+1} + \{(t\alpha_\lambda - \alpha_{\lambda-1}) - (t\alpha_\lambda - \alpha_\lambda)\}z^\lambda$$
$$+ (\alpha_{\lambda-1}-\alpha_{\lambda-2})z^{\lambda-1} + ... + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\}z + \alpha_0$$
$$+ i\{\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + \beta_0\}$$

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For $|z|>1$, we have by using the hypothesis

$$\begin{aligned}|F(z)| \ge &\left|a_n z^{n+1}+(kt-1)\alpha_n z^n\right|-\left|(kt\alpha_n-\alpha_{n-1})z^n+(t\alpha_{n-1}-\alpha_{n-2})z^{n-1}-(t-1)\alpha_{n-1}z^{n-1}\right.\\ &+......+(t\alpha_{\lambda+1}-\alpha_\lambda)z^{\lambda+1}-(t-1)\alpha_{\lambda+1})z^{\lambda+1}+(t\alpha_\lambda-\alpha_{\lambda-1})z^\lambda-(t-1)\alpha_\lambda z^\lambda\\ &+(\alpha_{\lambda-1}-\alpha_{\lambda-2})z^{\lambda-1}+......+\{(\alpha_1-\tau\alpha_0)+(\tau\alpha_0-\alpha_0)\}z+\alpha_0\\ &\left.+i\{\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+\beta_0\}\right|\\ \ge &|z|^n\Big[\ |a_n z+(kt-1)\alpha_n|-\Big\{|kt\alpha_n-\alpha_{n-1}|+\frac{|t\alpha_{n-1}-\alpha_{n-2}|}{|z|}+\frac{|t-1||\alpha_{n-1}|}{|z|^2}+......\\ &+\frac{|t\alpha_{\lambda+1}-\alpha_\lambda|}{|z|^{n-\lambda-1}}+\frac{|t-1||\alpha_{\lambda-1}|}{|z|^{n-\lambda-1}}+\frac{|t\alpha_\lambda-\alpha_{\lambda-1}|}{|z|^{n-\lambda}}+\frac{|t-1||\alpha_\lambda|}{|z|^{n-\lambda}}+\frac{|\alpha_{\lambda-1}-\alpha_{\lambda-2}|}{|z|^{n-\lambda+1}}\\ &+......+\frac{|\alpha_1-\tau\alpha_0|}{|z|^{n-1}}+\frac{(1-\tau)|\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}+\sum_{j=1}^{n}\frac{|\beta_j-\beta_{j-1}|}{|z|^{n-j}}+|\beta_0|\ \Big\}\ \Big]\\ \ge &|z|^n\Big[\ |a_n z+(kt-1)\alpha_n|-\{\ kt\alpha_n-\alpha_{n-1}+t\alpha_{n-1}-\alpha_{n-2}+(t-1)|\alpha_{n-1}|+......\\ &+t\alpha_{\lambda+1}-\alpha_\lambda+(t-1)|\alpha_{\lambda+1}|+t\alpha_\lambda-\alpha_{\lambda-1}+(t-1)|\alpha_\lambda|+\alpha_{\lambda-1}-\alpha_{\lambda-2}+......\\ &+\alpha_1-\tau\alpha_0+(1-\tau)|\alpha_0|+|\alpha_0|+\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)+|\beta_0|\ \}\ \Big]\\ \ge &|z|^n\Big[\ |a_n z+(kt-1)\alpha_n|-\{kt\alpha_n+2|\alpha_0|-\tau(\alpha_0+|\alpha_0|)+(t-1)\sum_{j=\lambda}^{n-1}(\alpha_j+|\alpha_j|)\\ &+2\sum_{j=0}^{n}|\beta_j|\ \}\ \Big]\\ >&\,0\end{aligned}$$

if

$$\begin{aligned}|a_n z+(kt-1)\alpha_n| > \{&kt\alpha_n+2|\alpha_0|-\tau(\alpha_0+|\alpha_0|)+(t-1)\sum_{j=\lambda}^{n-1}(\alpha_j+|\alpha_j|)\\ &+2\sum_{j=0}^{n}|\beta_j|\ \}\end{aligned}$$

This shows that the zeros of F(z) of modulus greater than 1 lie in

$$\left|z+(kt-1)\frac{\alpha_n}{a_n}\right| \le \frac{kt\alpha_n+2|\alpha_0|-\tau(\alpha_0+|\alpha_0|)+(t-1)\sum_{j=\lambda}^{n-1}(\alpha_j+|\alpha_j|)+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}.$$

Since the zeros of F(z) less than or equal to 1 also satisfy the above inequality, it follows that all the zeros of F(z) lie in

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$$\left| z+(kt-1)\frac{\alpha_n}{a_n} \right| \le \frac{kt\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + (t-1)\sum_{j=\lambda}^{n-1}(\alpha_j + |\alpha_j|) + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|} \quad .$$

Since the zeros of P(z) are als the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left| z+(kt-1)\frac{\alpha_n}{a_n} \right| \le \frac{kt\alpha_n + 2|\alpha_0| - \tau(\alpha_0 + |\alpha_0|) + (t-1)\sum_{j=\lambda}^{n-1}(\alpha_j + |\alpha_j|) + 2\sum_{j=0}^{n}|\beta_j|}{|a_n|} \quad .$$

That proves the result.

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